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# New Technique for solving parabolic equation with Natural growth in Musielak spaces

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**Abstract:**
In this paper, we study the existence of solutions for nonlinear parabolic equations with natural growth terms in Musielak spaces
$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = \text{f in } Q$
where $A(u)$ =- div $a(x,t,u,\nabla u)$, is a Leary lions operator type, $g(x,t,u,\nabla u)$ is a nonlinear with the sing condition but any restriction on its growth $u$ ,$H(x,t,\nabla u)$ is only growing at most at $|\nabla u|$

**Key words and phrases** : Musielek-Orlics Spaces, nonlinear parabolic equation,natural growth terms.

1 Introduction:
Let $\Omega$ be a bounded open subset of $R^N$, $N \geq 2$, and let $Q$ be the cylinder $(0,T) \times \Omega, T > 0$, we consider nonlinear parabolic problem

$$(P) = \begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f & in\ Q \\ u(x,0) = u_0(x) & in\ \Omega \\ u(x,t) = 0 & on \sum = \partial\Omega \times (0,T) \end{cases} \quad (1.1)$$

The study of nonlinear partial differential equations, particularly parabolic equations, is a vital field in mathematical analysis and applied physics. This is due to their central role in modeling phenomena that involve diffusion and temporal evolution, such as heat transfer, chemical diffusion, and the dynamics of non-Newtonian fluids[11,12] . Historically, efforts have focused on proving the existence of solutions for these equations within the classical Sobolev spaces $W^{1,p}(\Omega)$ [1].
However, many complex physical problems necessitate non-uniform or variable growth conditions, which has driven researchers to generalize the mathematical framework. The first major development was the transition to **Orlicz-Sobolev spaces** $W^{1,L_M}(\Omega)$ [19], which accommodated more general growth conditions[14,18] . Despite the importance of these spaces, previous studies within this context, such as the works of Donaldson[14] and Elmahi and Meskine[17,18 ], often assumed restrictive growth conditions on the N-function $M$ (such as the $\Delta_2$ or $\Delta_2'$ conditions), or excluded strong nonlinear terms.
More recently, with the need to handle highly non-uniform growth, the focus has shifted to the more comprehensive framework of **Musielak-Orlicz spaces** $W^{1,x}L_\varphi(Q)$ [24]. This framework offers a significant advantage as it naturally encompasses Orlicz-Sobolev spaces and Variable Exponent Lebesgue spaces [15,26]. Nevertheless, prior work in this setting typically assumed the absence of strong nonlinearities (i.e., $g = H = 0$) [3].

## Research Problem and Contribution of the Current Study

This paper addresses the existence of solutions for the strongly nonlinear parabolic equation (Problem (P)) within the general framework of **inhomogeneous Musielak-Orlicz spaces** $W^{1,x}L_\varphi(Q)$:

$$\frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f \quad \text{in } Q$$

where $A(u) = -\text{div } a(x,t,u,\nabla u)$ represents a Leray-Lions type operator [20,23] . The significant analytical difficulty and research challenge lie in the simultaneous inclusion of the two strong nonlinear terms $g(x,t,u,\nabla u)$ and $H(x,t,\nabla u)$ within this generalized framework.

**The main contribution of this study** is to establish the existence of solutions for Problem (P) in Musielak-Orlicz spaces, assuming that the nonlinear terms $g$ and $H$ satisfy **natural growth conditions** with respect to $u$ and $\nabla u$. Specifically:

1. **The term $g$:** is nonlinear and satisfies a sign condition, but is **without any restriction on its growth with respect to** $u$, which represents a significant generalization over studies that impose strict growth limitations.
2. **The term $H$:** grows at most at the rate of $|\nabla u|$.

This achievement constitutes a **major generalization** of the results by Elmahi and Meskine [17,18] , which were confined to Orlicz-Sobolev spaces, to the more general and flexible framework of Musielak-Orlicz spaces. To accomplish this, the study relies on a proof technique involving advanced A Priori Estimates and the use of convergence properties in Musielak-Orlicz spaces, thereby guaranteeing the existence of a solution $u \in D(A) \cap W^{1,x}L_{\varphi}(Q) \cap C(([0,T], L^2(\Omega)))$ that satisfies the Energy Equality for the problem.

**Structure of the Paper**

The paper is divided into several organized sections. Section 2 (BASIC ASSUMPTIONS) introduces the fundamental definitions and properties related to Musielak-Orlicz functions, Lebesgue and Musielak-Orlicz-Sobolev spaces, and natural growth conditions. Section 3 (Essential Result) presents the specific assumptions on the operator $A$ and the terms $g$ and $H$, and states the main existence theorem (Theorem 3.1). The remainder of the paper is dedicated to providing the detailed proof of the theorem, which includes the steps for A Priori Estimates, the almost everywhere convergence of gradients, and the use of the truncation technique to establish the existence of the solution.

## 2 Basic Assumptions

In this section, we introduce some definitions and known facts about Musielak-Orlicz-Sobolev spaces. Standard reference

### 2.1 Natural growth

In the context of strongly nonlinear parabolic equations, the term "Natural Growth Condition" refers to the maximal growth rate allowed for the nonlinear term $g(x,t,u,\nabla u)$ with respect to the gradient $\nabla u$ such that the existence of a weak solution can be established via standard energy estimates. Specifically, the condition is imposed as:

$$|g(x,t,u,\nabla u)| \le \rho(|u|)(c_2(x,t) + \varphi(x,|\nabla u|))$$

where $\varphi$ is the Musielak-Orlicz function, $\rho$ is a continuous, non-decreasing function, and $c_2 \in L^1(Q)$. This condition ensures that the growth of $g$ is controlled by the modular function of the underlying Musielak-Orlicz space.

### 2.2 Musielak-Orlicz function

Let $\Omega$ be an open subset of $IR^N$ $(N \ge 2)$, and let $\varphi(x,t)$ be a real-valued function defined in $\Omega \times IR^+$ and satisfying the following conditions:

**$(a)$ $\varphi(x,\cdot)$ is an $N$-function, *i.e.* convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$, and :**

$$\limsup_{t\to 0} \sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0 \quad , \quad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$$

**$(b)$ $\varphi(\cdot,t)$ is a measurable function.**

A function $\varphi(x,t)$ which satisfies conditions $(a)$ and $(b)$ is called a Musielak-Orlicz function. For a Musielak-Orlicz function $\varphi(x,t)$ we set $\varphi_x(t) = \varphi(x,t)$ and let $\varphi_x^{-1}(t)$ the reciprocal function with respect to $t$ of $\varphi_x(t)$, i.e.

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t.$$

For any two Musielak-Orlicz functions $\varphi(x,t)$ and $\gamma(x,t)$, we introduce the following ordering:

$(c)$ **If there exists two positives constants $c$ and $T$ such that for almost everywher $x \in \Omega$ :**

$$\varphi(x,t) \leq \gamma(x,ct) \quad \text{for} \quad t \geq T,$$

we write $\varphi \prec \gamma$, and we say that $\gamma$ dominate $\varphi$ globally if $T = 0$, and near infinity if $T > 0$.

$(d)$ **For every positive constant $c$ and almost everywhere $x \in \Omega$, if**

$$\lim_{t\to 0}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0 \quad or \quad \lim_{t\to\infty}(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}) = 0,$$

we write $\varphi \prec\prec \gamma$ at 0 or near $\infty$ respectively, and we say that $\varphi$ increases essentially more slowly than $\gamma$ at 0 or near $\infty$ respectively.

Let

$$\psi(x,s) = \sup_{t\geq 0}\{st - \varphi(x,t)\}$$

that is, $\psi(x,t)$ is the Musielak-Orlicz function complementary to (or conjugate) of $\varphi(x,t)$ in the sense of Young with respect to the variable $s$.

The Musielak-Orlicz function $\varphi(x,t)$ is said to satisfy the $\Delta_2$ −condition if, there exists $k > 0$ and a nonnegative function $h(\cdot) \in L^1(\Omega)$, such that

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \qquad \text{a.e.} \quad x \in \Omega,$$

for large values of $t$, or for all values of $t$.

## 2.3 Musielak-Orlicz Lebesgue space

In the following, the measurability of a function $u: \Omega \mapsto IR$ means the Lebesgue measurability. We define the functional

$$\varrho_{\varphi,\Omega}(u) = \int_\Omega \varphi\,(x, |u(x)|)\, dx$$

where $u: \Omega \mapsto IR$ is a measurable function. The set

$$K_\varphi(\Omega) = \{u: \Omega \mapsto IR \quad \text{measurable} \ / \ \varrho_{\varphi,\Omega}(u) < +\infty\}$$

is called the Musielak-Orlicz class (the generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz space) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$; equivalently

$$L_\varphi(\Omega) = \left\{u: \Omega \mapsto IR \quad \text{measurable} \ / \ \varrho_{\varphi,\Omega}(\frac{|u(x)|}{\lambda}) < +\infty, \quad \text{for some } \lambda > 0\right\}.$$

In the space $L_\varphi(\Omega)$, we define the following two norms:

$$||u||_{\varphi,\Omega} = \inf\left\{\lambda > 0 \ / \ \int_\Omega \varphi\,(x, \frac{|u(x)|}{\lambda})\, dx \leq 1\right\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{||v||_\psi \leq 1} \int_\Omega |\,u(x)v(x)|\, dx,$$

where $\psi(x,t)$ is the Musielak-Orlicz function complementary (or conjugate) to $\varphi(x,t)$. These two norms are equivalent [24] .

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is separable space and $E_\psi(\Omega)^* = L_\varphi(\Omega)$ [24].

The following conditions are equivalent:

$(e) E_\varphi(\Omega) = K_\varphi(\Omega)$

$(f) K_\varphi(\Omega) = L_\varphi(\Omega)$

$(g) \varphi$ has the $\Delta_2$ property.

We recall that $\varphi$ has the $\Delta_2$ property if there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h , integrable in $\Omega$ such that $\varphi(x,2t) \leq k\varphi(x,t) + h(x)$ for large values of t,or for all values of t ,according to whether$\Omega$ has finite measure or not.

## 2.4 Musielak-Orlicz-Sobolev space

Let us define the modular convergence: we say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0$$

For any fixed nonnegative integer m we define

$$W^m L_\varrho(\Omega) = \{U \in L\varphi(\Omega) : \forall|\alpha| \le m \ \ D^\alpha u \in L_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with nonnegative integers $\alpha_i$, $|\alpha| = |\alpha_1| + |\alpha_2| + \dots + |\alpha_n|$ and $D^\alpha u$ denotes the distributional derivatives. The space $W^m L_\varrho(\Omega)$ is called the Musielak-Orlicz-Sobolev space. Now, the functional

$$\bar{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\le m} \varrho_{\varphi,\Omega}(D^\alpha u) \quad and \quad ||u||^m_{,\varphi,\Omega} = \inf\{\lambda > 0 : \bar{\varrho}_{\varphi,\Omega}(\frac{u}{\lambda}) \le 1\}$$

for $u \in W^m L_\varphi(\Omega)$, these functional are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively. The pair $\langle W^m L_\varphi(\Omega), ||u||^m_{,\varphi,\Omega}\rangle$ is a Banach space if $\varphi$ satisfies the following condition :

there exist a constant $c > 0$ such that $\inf_{x\in\Omega} \varphi(x,1) \ge c$.

as in [24] .

The spaces $W^m L_\varphi(\Omega)$ will always be identified to a $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed subspace of the product $\Pi_{\{|\alpha|\le m\}} = L_\varphi(\Omega) = \Pi L_\varphi$

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$

Let $W^m E_\varphi(\Omega)$ be the space of functions u suth that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$ , and let $W_0^m L_\varphi(\Omega)$ be the (norm) closure of the Schwartz space D(Ω) in $W^m E_\varphi(\Omega)$

The following spaces of distribution will also be used :

$$W^{-m} L_\psi(\Omega) = \{f \in D'(\Omega);\ \ f = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} D^\alpha f_\alpha \ \ with \ \ f_\alpha \in L_\psi(\Omega)\}$$

and

$$W^{-m} E_\psi(\Omega) = \{f \in D'(\Omega);\ \ f = \sum_{|\alpha|\le 1} (-1)^{|\alpha|} D^\alpha f_\alpha \ \ with \ \ f_\alpha \in E_\psi(\Omega)\}$$

As we did for $L_\varphi(\Omega)$,we say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\varrho}_{\varphi,\Omega}(\frac{u_n - u}{k}) = 0$$

From, for twe complementary Musielak-Orlicz functions $\varphi$ and $\psi$ the follwing

$h$) the young inequality:

$$t.s \le \varphi(x,t) + \psi(x,s) \ fort, s \ge 0, x \in \Omega$$

$i$)the Holder inequality :

$$\left|\int_\Omega u(x)v(x)dx\right| \le \| u \|_{\varphi,\Omega} \| v \|_{\varphi,\Omega}$$

for all $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$.

## 2.5 Inhomogeneous Musielak-Orlicz-Sobolev space

LetΩ an bounded open subset of $R^N$ and let $Q = \Omega \times ]0,T[$ with some given $T > 0$.Let $\varphi$ be a Musielak function .For each $\alpha \in N^n$, denote by $D_x^\alpha$ the distributional Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x} L_\alpha(Q) = U \in L_\alpha(Q) : \forall|\alpha| \le 1 D_x^\alpha u \in L_\varphi(Q)$$

and

$$W^{1,x} E_\alpha(Q) = U \in E_\alpha(Q) : \forall|\alpha| \le 1 D_x^\alpha u \in E_\varphi(Q)$$

The last space is a subspace of the first one ,and both are Banach spaces under the norm

$$||u|| = \sum_{|\alpha|\le m} ||D_x^\alpha u||_{\varphi,\Omega}$$

**Proposition 2.1.**. If $u \in L_\alpha(Q)$ then $u_\mu$ is measurable in Q and $\frac{\partial u_n}{\partial t} = \mu(u - u_\mu)$ and if $u \in L_\alpha(Q)$ then

$$\int_Q \varphi(x, u_\mu) dxdt \le \int_Q \varphi(x,u) dxdt.$$

**Proposition 2.2.**(1)If $u \in L_\alpha(Q)$ then $u_\mu \to u$ as $\mu \to \infty$ in $L_\alpha(Q)$ for the modular convergence. (2)If $u \in W^{1,x}L_\alpha(Q)$ then $u_\mu \to u$ as $\mu \to \infty$ in $W^{1,x}L_\alpha(Q)$ for the modular convergence.
**Remark 2.1.** If $u \in E_\varphi(Q)$, we can choose $\lambda$ arbitrary small since $D(Q)$ is (norm) dense in $E_\varphi(Q)$. thuse , for all $\lambda > 0$

$$\int_Q \varphi\,(x, \frac{u_\mu - u}{\lambda}) dxdt \to 0 as \mu \to \infty$$

and $\mu \to u$strongly in $E_\varphi(Q)$.Idem for $W^{1,x}L_\alpha(Q)$.

## 3 Essential Result

let $\Omega$ be a bounded open set of $R^N (N \geq 1)$,$T > 0$ is given and we set $Q_T = \Omega \times (0,T)$
Throughout this section, we denote $Q_\tau = \Omega \times (0,\tau)$ for every $\tau \in [0,T]$. Let $\varphi$ and $\gamma$ two Musiclak-Orlicz functions such that $\gamma \ll \varphi$. Consider a second-order operator $A: D(A) \subset W^{1,x}L_\varphi(Q) \to W^{-1,x}L\psi(Q)$ of the form

$$A(u) = -\text{div}\ a(x,t,u,\nabla u)$$

where $a: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, for almost every$(x,t) \in \Omega \times [0,T]$ and all $s \in \mathbb{R}, \xi \neq \xi^* \in \mathbb{R}^N$

$$|a(x,,t,s,\xi)| \leq \beta(c_1(x,t)) + \psi_x^{-1}\gamma(x,v|s|)) + \psi_x^{-1}\varphi(x,v|\xi|)), \tag{3.1}$$

$$(a(x,t,s,\xi) - a(x,t,s,\xi^*)) \cdot (\xi - \xi^*) > 0 \quad \text{for} \quad \xi \neq \xi^*, \tag{3.2}$$

$$a(x,t,s,\xi) \cdot \xi \geq \alpha \cdot \varphi(x, \tfrac{|\xi|}{\lambda}) - d(x,t) \tag{3.3}$$

with $c_1(x,t) \in E_\psi(Q), c_1 \geq 0, d(x,t) \in L^1(Q), \alpha, \beta, \theta > 0$ .
Assume that $g: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for almost. every $(x,t) \in \Omega \times [0,T]$ and for all $s \in \mathbb{R}, \xi \in \mathbb{R}^N$;

$$|g(x,t,s,\xi)| \leq \rho(|s|)(c_2(x,t) + \varphi(x,|\xi|)) \tag{3.4}$$

$$g(x,t,s,\xi)s \geq 0 \tag{3.5}$$

Furthermore , Let $g(x,t,s,\xi): Q_T \times R \times \mathbb{R}^N \to \mathbb{R}^N$ and with $c_2(x,t) \in L^1(Q)$ and $b: \mathbb{R}^+ \to \mathbb{R}^+$ is a contimuous and nondecreasing function. Furtheremore let

$$f \in W^{-1,x}E_\psi(Q) \tag{3.6}$$

Assume that $H: \Omega \times [0,T]: \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, for almost. every $(x,t) \in \Omega \times [0,T]$ :$\xi \in \mathbb{R}^N$;
The function $H(x,t,\nabla u)$ must be non-negative and bounded growth suth that

$$H(x,t,\nabla u) \leq C(1 + \varphi|(x,\nabla u)|) \tag{3.7}$$

$$H(x,t,\nabla u) \geq 0 \tag{3.8}$$

consider then the following parabolic initial-boundary value problem

$$(P) = \begin{cases} \dfrac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f & in\ Q \\ u(x,0) = u_0(x) & in\ \Omega \\ u(x,t) = 0 & on \sum = \partial\Omega \times (0,T) \end{cases} \tag{3.9}$$

where $u_0$ is a given function in $L^2(\Omega)$ .
Where shall prove the follwing existence theorem.
**Theorem(3.1)** Assume that (3.1)-(3.8) hold true.Then the problem (3.9) admits are solutions $u \in D(A) \cap W_0^{1,x}L_\varphi(Q) \cap C\big(([0,T], L^2(\Omega)\big)$ such that $g(x,t,u,\nabla u) \in L^1(Q), g(x,t,u,\nabla u)u \in L^1(Q)$ . Furthermore $u(x,0) = u_0(x)$ for almost every $x \in \Omega$, and for all $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$ with $\frac{\partial v}{\partial t} \in W^{-1,x}L\psi(Q) + L^2(Q)$ and for all $\tau \in [0,T]$, we have

$$\left\langle \frac{\partial v}{\partial t}, u \right\rangle_{Q_\tau} + \left[\int_\Omega u\,(t) v(t) dx\right]_0^\tau + \int_{Q_\tau} a\,(x,t,u,\nabla u)\nabla v dxdt + \int_{Q_\tau} g\,(x,t,u,\nabla u) v dxdt + \int_{Q_\tau} H\,(x,t,\nabla u) v dxdt = \langle f, v\rangle_{Q_\tau} \tag{3.10}$$

and for $v = u$, which gives the energy equality

$$\frac{1}{2}\int_{\Omega} u^2\,(\tau)dx - \frac{1}{2}\int_{\Omega} u_0^2\,dx + \int_{Q_\tau} a\,(x,t,u,\nabla u)\nabla u\,dxdt$$
$$+\int_{Q_\tau} g\,(x,t,u,\nabla u)vdxdt + \int_{Q_\tau} H\,(x,t,\nabla u)vdxdt = \langle f,u\rangle_{Q_\tau}$$

**Proof of Theorem(3.1)** We divide the proof in four steps.

**Step 1:A priori estimates**

Consider the sequence of approximate problems:

$$\begin{cases} u_n \in D(A)\cap W_0^{1,x}L_v(Q)\cap C(([0,T],L^2(\Omega)),u_n(x,0)=u_0(x)a.e.\in\Omega, \\ \left\langle\frac{\partial u_n}{\partial t},v\right\rangle + \langle A(u_n),v\rangle + \int_Q g_n\,(x,t,u_n,\nabla u_n)vdxdt + \int_Q H_n\,(x,t,\nabla u_n)vdxdt = \langle f,v\rangle \\ for\ all\ v\in W_0^{1,x}L_v(Q) \end{cases}$$

Let $(f_n)_{n\in N}\in W^{-1}E_\psi(\Omega)$ be a sequence of smooth functions such that $f_n\to f$ in $L^1(\Omega)$ and $|f_n|\le|f|$(for example $f_n=T_n(f)$)

where
$g_n(x,t,s,\xi)=T_n(g_n(x,t,s,\xi))$.
And

$$H_n(x,t,\nabla u_n)=\frac{H_n(x,t,\nabla u_n)}{1+H_n(x,t,\nabla u_n)}\ ,g_n(x,t,u_n,\nabla u_n)\frac{g_n(x,t,u,\nabla u_n)}{1+g_n(x,t,u,\nabla u_n)}$$

and where for $m>0,T_m$ means for the usual truncation operator at m defined on R by

$$T_m(s)=max(-m,min(k,s))$$

Note that $g_n(x,t,s,\xi)s\ge 0\,,|g_n(x,t,s,\xi)|\le|g(x,t,s,\xi)|\ and\ |g_n(x,t,s,\xi)|\le n\,,|H_n(x,t,\nabla u)|\le n$ . since $g_n$ is bounded for any fixed $n>0$, there exists at last one solution $u_n$ of (3,11), (the existence of $u_n$ can be obtained from Galerkin solutions corresponding to Equation (3,11) as in [21] . Note also that $\langle u_n',u_n\rangle$ is defined in the sense of distributions (where $u_n'=\frac{\partial u_n}{\partial t}$ means for the time derivative of $u_n$).Since $u_n'=f-A(u_n)-g_n-H_n$ is in $W^{-1,x}L_\psi(Q)$ we can exted $\langle u_n',u_n\rangle$ to all $u_n\in W_0^{1,x}L_v(Q)$. Using in(3,10) the test function $u_n$,we get

$$\frac{1}{2}\int_{\Omega} u_n^2\,(\tau)dx - \frac{1}{2}\int_{\Omega} u_0^2\,dx + \int_{Q_\tau} a\,(x,t,u_n,\nabla u_n)\nabla u_n dxdt + \int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)u_n dxdt$$
$$+\int_{Q_\tau} H_n\,(x,t,\nabla u_n)u_n dxdt = \langle f,u_n\rangle$$

The terms involving $u_n^2$ and $g_n$ ,$H_n$ were moved to the other side:

$$\int_{Q_\tau} a\,(x,t,u_n,\nabla u_n)\nabla u_n dxdt = \langle f,u_n\rangle - \frac{1}{2}\int_{\Omega} u_n^2\,(\tau)dx + \frac{1}{2}\int_{\Omega} u_0^2\,dx$$
$$-\int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)u_n dxdt - \int_{Q_\tau} H_n\,(x,t,\nabla u_n)u_n dxdt$$

Sine $g_n\ge 0$ ,$H_n\ge 0$ and $u_n^2(\tau)\ge 0$ ,the inequality was estimated as :

$$\int_Q a\,(x,t,u_n,\nabla u_n)\nabla u_n dxdt \le \langle f,u_n\rangle + \frac{1}{2}\int_{\Omega} u_0^2\,dx + c$$

The sum $\frac{1}{2}\int_\Omega u_0^2\,dx + c$ was replaced with the constant C for simplicity ,yielding:

$$\int_Q a\,(x,t,u_n,\nabla u_n)\nabla u_n dxdt \le \langle f,u_n\rangle + C$$

Where here and below C is a positive constant not depending on n. By theorem (3.1) of [5] we can say that:

$$\begin{cases} u_n \text{ is bounded in } W_0^{1,x} L_\varphi(Q), in \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt \le C, \\ \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)u_n dxdt \le C \text{ and } \int_{Q_\tau} H_n(x,t,\nabla u_n)u_n dxdt \le C \end{cases} \quad (3.12)$$

To prove that

$a(x,t,u_n,\nabla u_n)$ is a bounded sequence in $(L_\psi(Q))^N$. Let $\eta \in (E_\varphi(Q))^N$ With $||\eta||_{\varphi,Q} = 1$
In view of (3.2) Let $\xi = \nabla u_n$ and $\xi^* = \eta$

$$[a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\eta)][\nabla u_n - \eta] \ge 0.$$

we have

$$\int_Q [a(x,t,u_n,\nabla u_n) - a(x,t,u_n,\eta)][\nabla u_n - \eta]dxdt \ge 0.$$

which gives

$$\int_Q a(x,t,u_n,\nabla u_n)\eta dxdt \le \int_Q a(x,t,u_n,\nabla u_n)\nabla u_n dxdt + \int_Q a(x,t,u_n,\eta)[\nabla u_n - \eta]dxdt$$

Using (3.1) and (3.12),we easily see that

$$\int_Q a(x,t,u_n,\nabla u_n)\eta dxdt \le C$$

And so $a(x,t,u_n,\nabla u_n)$ is a bounded sequence in $(L_\psi(Q))^N$ . splitting Q into $|u_n| \le 1$ *and* $|u_n| \ge 1$ and using (3.4) we can write

$$\int_Q |g_n(x,t,u_n,\nabla u_n)|dxdt \le \rho(1) \int_{|u_n|\le 1} \big(c_2(x,t) + \varphi(|\nabla T_1(u_n)|)\big)dxdt$$
$$+ \int_{|u_n|\ge 1} g_n(x,t,u_n,\nabla u_n)u_n dxdt \le C$$

And then $g_n(x,t,u_n,\nabla u_n)$ is a bounded sequence in $L^1(Q)$ implying that $\frac{\partial u_n}{\partial t}$ is a bounded sequence in $W^{-1,x}L_\psi(Q) + L^1(Q)$,therefore Corollary 4.5 allows us to deduce that $u_n \to u$ strongly in $L^1(Q)$.Thus , for some subsequence still denoted by $u_n$ and for some $h \in (L_\psi(Q))^N$. :

$$\begin{cases} u_n \to u \text{ weakly in } W^{1,x}L_\varphi(Q) \text{ for} \sigma(\prod L_\varphi, \prod E_\psi), \text{ strongly in } L^1(Q) \\ \text{and a.e. in } Q \text{ and } a(x,t,u_n,\nabla u_n) \rightharpoonup h \text{ in } (L_\varphi(Q))^N \text{for } \sigma(\prod L_\varphi, \prod E_\psi). \end{cases} \quad (3.13)$$

**Step 2: convergence of gradients.**

Fix $m > 0$ and let $\psi(s) = s\, exp(\delta s^2)$, $\delta > 0$. It is well known that when $\delta \ge (\frac{\rho(m)}{2\alpha})^2$ one has

$$\psi'(s) - \frac{\rho(m)}{\alpha} \mid \psi(s) \mid \ge \frac{1}{2} \quad for\ all\ s \in R \qquad (3.14)$$

Let $v_j \in D(Q)$ be a sequence such that

$$v_j \to u \ \ in\ W_0^{1,x}L_\varphi(Q) for\ the\ modular\ convergence \qquad (3.15)$$

and let $\kappa_i \in D(\Omega)$ be a sequence which converges strongly to $u_0$ in $L^2(\Omega)$ .Set $\kappa_{\mu,j}^i = T_m(v_j)_\mu + exp(-\mu t)T_m(\kappa_i)$ where $T_m(v_j)\mu$ is the mollification with respect to time of $T_m(v_j)\mu$ . Note that $\kappa_{\mu,j}^i$ is smooth function having the following properties:

$$\begin{cases} \frac{\partial}{\partial t}(\kappa^i_{\mu,j}) = \mu(T_m(v_j) - \kappa^i_{\mu,j}), \kappa^i_{\mu,j}(0) = T_m(v_j), |\ \kappa^i_{\mu,j}\ | \leq m, \\ \kappa^i_{\mu,j} \to T_m(u)\mu + exp(-\mu t)T_m(\kappa_i)\ in\ W_0^{1,x}L_\varphi(Q) \\ for\ \ the\ modular\ convergence\ as\ j \to \propto, \\ T_m(u)_\mu + exp(-\mu t)T_m(\kappa_i) \to T_m(u)\ in\ W_0^{1,x}L_\varphi(Q) \\ for\ the\ modular\ convergence\ as\ \mu \to \propto. \end{cases}$$

using in (3.11) the test function $z^{\mu,i}_{n,j} = \psi(T_m(u_n) - \kappa^i_{\mu,j})$, which belongs to $W_0^{1,x}L_\varphi(Q)$ we get $\langle u'_n, z^{\mu,i}_{n,j}\rangle + \int_Q a\,(x,t,u_n,\nabla u_n)[\nabla T_m(u_n) - \nabla\kappa^i_{\mu,j}]\psi'(T_m(u_n) - \kappa^i_{\mu,j})\ dxdt$
$+\int_Q g_n\,(x,t,u_n,\nabla u_n)\psi(T_m(u_n) - \kappa^i_{\mu,j})\ dxdt+$

$$\int_Q H_n\,(x,t,u_n,\nabla u_n)\psi(T_m(u_n) - \kappa^i_{\mu,j})\ dx\ dt = \langle f, \psi(T_m(u_n) - \kappa^i_\mu)\rangle$$

which implies since $g_n(x,t,u_n,\nabla u_n)\psi\big(T_m(u_n) - \kappa^i_{\mu,j}\big) \geq 0$ on ${\arrowvert u_{n}\arrowvert > m}:$

$$\langle u'_n, z^{\mu,j,i}_n\rangle + \int_Q a\,(x,t,u_n,\nabla u_n)\big[\nabla T_m(u_n) - \nabla\kappa^i_{\mu,j}\big]\psi'\big(T_m(u_n) - \kappa^i_{\mu,j}\big)\ dx\ dt$$
$$+\int_{|u_n|\leq m} g_n\,(x,t,u_n,\nabla u_n)\psi(T_m(u_n) - \kappa^i_{\mu,j})\ dx\ dt$$
$$+\int_{|u_n|\leq m} H_n\,(x,t,\nabla u_n)\psi(T_m(u_n) - \kappa^i_{\mu,j})\ dx\ dt \leq \langle f, \psi(T_m(u_n) - \kappa^i_\mu)\rangle. \qquad (3.16)$$

In the sequel and throughout the paper, we will omit for simplicity to denote $x$ and $t$ in the function $a(x,t,s,\xi)$ and denote $\varepsilon(n,j,\mu,i,s)$ all quantities (possibly different) such that

$$\lim_{s\to\infty}\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,\mu,i) = 0$$

and this will be the order in which the parameters we use will tend to infinity,that is , first $n$ ,than $j,\mu,i$, and finally $s$. Similarly we will write only $\epsilon(n)$, or $\epsilon(n,j)$, ... to mean that the limits are made only on the specified parameters. We will deal with each term of (1.16). First of all, observe that

$$\langle\ f\ ,\ \psi(T_m(u_n) - \kappa^i_{\mu,j})\rangle = \epsilon(n,j,\mu), \qquad (3.17)$$

since $(T_m(u_n) - \kappa^i_{\mu,j}) \rightharpoonup (T_m(u) - \kappa^i_{\mu,j})$ weakly in $W_0^{1,x}L_\varphi(Q)$ as $n \to \infty$, and $T_m(u) - \kappa^i_{\mu,j} \to T_m(u) - T_m(u)_\mu + exp(-\mu t)T_m(\kappa_i)$ in $W_0^{1,x}L_\varphi(Q)$ for the modular convergence and so for the topology $\sigma(\prod L_\varphi, \prod L_\psi)$ as $j \to +\infty$ .
From (3.11)one deduces that $u_n \in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial u_n}{\partial t} \in W^{-1,x}L_\psi(Q)$ and than , by Theorem (3.2) there exists a smooth function $u_{n\sigma}$ such that $\sigma \to \infty, u_{n,\sigma} \to u_n$ in $\in W_0^{1,x}L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t}$ in $W^{-1,x}L_\psi(Q) + L^2(Q)$ for the modular convergence . Consequently

$$\langle\frac{\partial u_n}{\partial t}, z^{\mu,i}_{n,j}\rangle = \lim_{\sigma\to\infty}\int_Q \frac{\partial u_{n\sigma}}{\partial t}\ \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\ dx\ dt$$
$$= \lim_{\sigma\to\infty}\int_Q [\ (T_m(u_{n\sigma}))' + (G_m(u_{n\sigma}))']\ \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\ dx\ dt,$$

where $G_m(s) = s - T_m(s)$. Hence

$$\langle\frac{\partial u_n}{\partial t}, z^{\mu,i}_{n,j}\rangle = \lim_{\sigma\to\infty}\int_Q (\,T_m(u_{n\sigma}) - \kappa^i_{\mu,j})'\psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\ dx\ dt$$
$$+\int_Q (\,\kappa^i_{\mu,J})'\ \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\ dx\ dt$$
$$+\int_Q (\,G_m(u_{n\sigma}))'\ \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\ dx\ dt$$
$$= \lim_{\sigma\to\infty}(I_1(\sigma) + I_2(\sigma) + I_3(\sigma)).$$

Setting $\psi(s) = \int_0^s \eta(r)\, dr$, it is easy to see that, since $\eta(s) \geq 0$

$$I_1(\sigma) = [\int_\Omega \eta\,(T_m(u_{n\sigma})(t) - \kappa^i_{\mu,j}) dx]_0^T$$
$$\geq -\int_\Omega \eta\,(T_m(u_{n\sigma})(0) - T_m(\kappa_i))\, dx$$

and Since, as $\sigma \to \infty$ the last side goes to

$$-\int_\Omega \eta\,(T_m(u_0) - T_m(\kappa_i))\, dx,$$

which is of the from $\varepsilon(i)$,we get
$\limsup_{\sigma\to\infty} I_1(\sigma) \geq \varepsilon(i)$.
About $I_2(\sigma)$, we have, since $(\kappa^i_{\mu,j})' = \mu(T_m(v_j) - \kappa^i_{\mu,j})$ and $\psi(s)s \geq 0$

$$I_2(\sigma) = \mu \int_Q (T_m(v_j) - \kappa^i_{\mu,j})\, \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\, dx\, dt$$
$$\geq \mu \int_Q (T_m(v_j) - T_m(u_{n\sigma}))\psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})\, dx\, dt.$$

Since, as $\sigma \to \infty$, ,the last side goes to

$$\mu \int_Q (T_m(v_j) - T_m(u_n))\psi(T_m(u_n) - \kappa^i_{\mu,j})\, dx\, dt,$$

which is of form $\varepsilon(n, j)$, we obtain

$$\limsup_{\sigma\to\infty} I_2(\sigma) \geq \varepsilon(n, j).$$

For what concerns $I_3(\sigma)$,one has by integration by parts

$$I_3(\sigma) = -\int_Q G_m\,(u_{n\sigma})\, \psi'(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})'\, dx\, dt$$
$$+[\int_\Omega G_m\,(u_{n\sigma})(t)\, \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})(t)\, dx]_0^T$$

since $(T_m(u_{n\sigma}))' = 0$ on $\{|u_{n\sigma}| > m\}$ and $G_m(u_{n\sigma}) = 0$ on$\{|u_{n\sigma}| \leq m\}$ . since

$$[\int_\Omega G_m\,(u_{n\sigma})(t)\, \psi(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})(t)\, dx]_0^T \geq -\int_\Omega G_m\,(u_{n\sigma})(0)\psi(T_m(u_{n\sigma})(0)$$
$$-T_m(\kappa_i)\, dx$$

we have

$$I_3(\sigma) \geq \int_Q G_m\,(u_{n\sigma})\, \psi'(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})(\kappa^i_{\mu,j})'\, dxdt$$
$$-\int_\Omega G_m\,(u_{n\sigma}(0))\, \psi(T_m(u_{n\sigma}(0)) - T_m(\kappa_i))\, dx,$$
$$= \mu \int_Q G_m\,(u_{n\sigma})\, \psi'(T_m(u_{n\sigma}) - \kappa^i_{\mu,j})(T_m(v_j) - \kappa^i_{\mu,j})\, dxdt$$
$$-\int_\Omega G_m\,(u_{n\sigma}(0))\, \psi(T_m(u_{n\sigma}(0)) - T_m(\kappa_i))dx,$$

which implies that

$$\limsup_{\sigma\to\infty} I_3(\sigma) \geq \mu \int_Q G_m\,(u_n)\, \psi'(T_m(u_n) - \kappa^i_{\mu,j})(T_m(v_j) - \kappa^i_{\mu,j})\, dx\, dt$$
$$-\int_\Omega G_m\,(u_0)\, \psi(T_m(u_0) - T_m(\kappa_i))\, dx$$

and hence ,by lettingas $n \to \infty$ is the integral of last side,

$$\limsup_{\sigma\to\infty} I_3(\sigma) \geq \mu \int_Q G_m\,(u)\, \psi'(T_m(u) - \kappa^i_{\mu,j})(T_m(v_j) - \kappa^i_{\mu,j})\, dx\, dt$$
$$-\int_\Omega G_m\,(u_0)\, \psi(T_m(u_0) - T_m(\kappa_i))\, dx + \varepsilon(n),$$
$$\geq \mu \int_\Omega G_m\,(u)\, \psi'(T_m(u) - \kappa^i_{\mu,j})(T_m(v_j) - T_m(u))\, dxdt$$

$$-\int_{\Omega} G_m(u_0)\, \psi(T_m(u_0) - T_m(\kappa_i))\, dx + \varepsilon(n), \qquad (3.18)$$

where we have used the fact that (recall that $|\kappa^i_{\mu,j}| \leq m$ )

$$\int_{\Omega} G_m(u)\, \psi'(T_m(u) - \kappa^i_{\mu,j})(T_m(u) - \kappa^i_{\mu,j})\, dx\, dt$$
$$= \int_{\{u>m\}} (u - m)\, \psi'(m - \kappa^i_{\mu,j})(m - \kappa^i_{\mu,j})\, dx\, dt$$
$$+ \int_{\{u<-m\}} (u + m)\, \psi'(-m - \kappa^i_{\mu,j})(-m - \kappa^i_{\mu,j})\, dx\, dt \geq 0$$

Since the first integral of last side of (3.18) is of the form $\varepsilon(j)$ while the second one is of the form $\varepsilon(i)$ we deduce that

$$\limsup_{\sigma\to\infty} I_3(\sigma) \geq \epsilon(n, j, i)$$

. combining these estimates on each $L_i$, we get

$$\langle(\frac{\partial u_n}{\partial t}), \psi(T_m(u_n) - \kappa^i_{\mu,j})\rangle \geq \varepsilon(n, j, i). \qquad (3.19)$$

For $s > 0$ ,set $Q^s = \{(x,t) \in Q: |\nabla T_m(u)| \leq s\}$ and $Q^s_j = \{(x,t) \in Q: |\nabla T_m(v_j)| \leq s\}$ and denote by $\chi^s$ and $\chi^s_j$ the characteristic function of $Q^s$ and $Q^s_j$,respectively On other hand,the second term of the left-hand side of (3.16) reads as

$$\int_Q a(x,t,u_n,\nabla u_n)[\nabla T_m(u_n) - \nabla\kappa^i_{\mu,j}]\psi'(T_m(u_n) - v^i_{\mu,j})\, dx\, dt$$
$$= \int_Q a(x,t,T_m(x,t,u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(v_j)\chi^s_j)[\nabla T_m(u_n) - \nabla T_m(v_j)\chi^s_j]$$
$$\times \psi'(T_m(u_n) - \kappa^i_{\mu,j})\, dx\, dt$$
$$+ \int_Q a(x,t,T_m(u_n),\nabla T_m(v_j)\chi^s_j)[\nabla T_m(u_n) - \nabla T_m(v_j)\chi^s_j] \times \psi'(T_m(u_n) - \kappa^i_{\mu,j}) dxdt$$
$$+ \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(v_j)\chi^s_j\psi'(T_m(u_n) - \kappa^i_{\mu,j}) dxdt$$
$$- \int_Q a(x,t,u_n,\nabla u_n)\nabla\kappa^i_{\mu,j}\psi'(T_m(u_n) - \kappa^i_{\mu,j})\, dx\, dt$$
$$:= J_1 + J_2 + J_3 + J_4$$

We shall go the limit as $n, j, \mu$ and $s \to \infty$ in the last three integrals of the last side. staring with $j_2$, we have by letting $s \to \infty$

$$J_2 = \int_Q a(T_m(u_n),\nabla T_m(v_j)\chi^s_j)[\nabla T_m(u_n) - \nabla T_m(v_j)\chi^s_j]\psi'(T_m(u_n) - \kappa^i_{\mu,j}) dxdt + \varepsilon(n)$$

sine $a(T_m(u_n),\nabla T_m(v_j)\chi^s_j) \to a(T_m(u),\nabla T_m(v_j)$ strongly in $(E_\psi(Q))^N$ by using (3.1) and Lebesgue theorem while $\nabla T_m u_n \rightharpoonup \nabla T_m(u)$weakly in $(L_\Phi(Q))^N$ BY (3.13).Letting $j \to \infty$ in the firt term of last side of the above equality ,one has ,sine $a(x,t,T_m(u_n),\nabla T_m(v_j)\chi^s_j) \to a(x,t,T_m(u),\nabla T_m(v)\chi^s$ strongly in $(E_\psi(Q))^N$ by using (3.1),(3.15) and lebesgue theorem while $\nabla T_m(v_j)\chi^s_j \to \nabla T_m(v_j)\chi^s$ strongly in $(L_\Phi(Q))^N$

$$J_2 = \int_{Q/Q^s} a(x,t,T_m(u_n),0)\nabla T_m(u)\psi'(T_m(u) - T_(u)_\mu - exp(-\mu t)T_m(\kappa_i)) dxdt + \varepsilon(n,j)$$

sine $\psi'((T_m(u) - T_m(u)_\mu - exp(-\mu t)T_m(\kappa_i)) \to 1$ a.e in $Q$ and is uniformly bounded by $\psi'(2m)$ we can let $\mu \to \infty$ in the first term of the last side to get

$$J_2 = \int_{Q/Q^s} a(x,t,T_m(u_n),0)\nabla T_m(u) dxdt + \varepsilon(n,j,\mu)$$

and thus ,by letting $s \to \infty$,we conclude that $J_2 = \varepsilon(n,j,\mu,s)$

$$J_3 = \int_{\{|u_n|\leq m\}} a(x,t,u_n,\nabla u_n)\nabla T_m(v_j)\chi^s_j\psi'\big(T_m(u_n) - \kappa^i_{\mu,j}\big) dxdt$$
$$\int_{\{|\nabla u_n|>m\}} a(x,t,T_m(u_n),0)\nabla T_m(v_j)\chi^s_j\psi'(T_m(u_n) - \kappa^i_{\mu,j}) dxdt$$

which given by letting $n \to \infty$,thanks to (3.13)

$$J_3 = \int_{\{|\nabla u_n| \le m\}} h \nabla T_m(v_j) \chi^s \, \psi'\big(T_m(u_n) - \kappa^i_{\mu,j}\big) dxdt$$

$$+ \int_{\{|\nabla u_n| > m\}} a\,(x,t,T_m(u_n),0) \nabla T_m(v_j) \chi^s_j \psi'(T_m(u_n) - \kappa^i_{\mu,j}) dxdt + \varepsilon(n)$$

so that letting $j \to \infty$ in two first integrals last of the last side and using (1.15),

$$J_3 = \int_{\{|\nabla u_n| \le m\}} h \nabla T_m(v_j) \chi^s \, \psi'\big(T_m(u) - T_m(u)_\mu - exp(-\mu t) T_m(\kappa_i) dxdt + \varepsilon(n,j)\big)$$

in which we can let $\mu \to \infty$ to obtain

$$J_3 = \int_Q h \, \nabla T_m(u) \chi^s dxdt + \varepsilon(n,j,\mu)$$

Consequently by letting $s \to \infty$

$$J_3 = \int_Q h \, \nabla T_m(u) dxdt + \varepsilon(n,j,\mu,s)$$

For what concerns $J_4$ we have ,as above ,by letting first $n$ than $j$ and finally $\mu$ go to infinity:

$$J_4 = \int_Q h \, \nabla \kappa^i_{\mu,j} \, \psi'(T_m(u) - \kappa^i_{\mu,j}) dxdt + \varepsilon(n)$$

$$= \int_Q h \, [\nabla T_m(u)_\mu - exp(-\mu t) T_m(\kappa_i)]$$

$$\psi'(T_m(u) - T_m(u)_\mu - exp(-\mu t) T_m(\kappa_i) dxdt + \varepsilon(n,j))$$

$$= -\int_Q h \, \nabla T_m(u) dxdt + \varepsilon(n,j,\mu)$$

We conclude than that

$$\int_Q a\,(x,t,u_n,\nabla u_n)[\nabla T_m(u_n) - \nabla w^i_{\mu,j}] \, \psi'(T_m(u_n) - \kappa^i_{\mu,j}) \, dx \, dt$$

$$= \int_Q [\, a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(T_m(u_n),\nabla T_m(v_j)\chi^s_j)][\nabla T_m(u_n) - \nabla T_m(v_j)\chi^s_j]$$

$$\times \psi'(T_m(u_n) - \kappa^i_{\mu,j}) \, dx \, dt + \varepsilon(n,j,\mu,s) \qquad (3.20)$$

The third term of the left-hand side of (3.16) can be estimated as

$$\left| \iint_{|u_n| \le m} g_n\,(x,t,u_n,\nabla u_n) \psi\big(T_m(u_n) - \kappa^i_{\mu,j}\big) \, dx \, dt \right|$$

$$+ \left| \iint_{|u_n| \le m} H_n\,(x,t,,\nabla u_n) \psi\big(T_m(u_n) - \kappa^i_{\mu,j}\big) \, dx \, dt \right|$$

$$\le \rho(m) \int_Q \left( c_2(x,t) + \frac{1}{\alpha} d(x,t) \right) |\psi(T_m(u_n) - \kappa^i_{\mu,j})| \, dx \, dt$$

$$+ \frac{\rho(m)}{\alpha} \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n)) \, \nabla T_m(u_n) |\psi(T_m(u_n) - \kappa^i_{\mu,j})| \, dx \, dt \qquad (3.21)$$

since $c_2(x,t)$ and $d(x,t)$ belongs to $L^1(Q)$ it is easy to see that

$$\rho(m) \int_Q \left( c_2(x,t) + \frac{1}{\alpha} d(x,t) \right) |\psi(T_m(u_n) - \kappa^i_{\mu,j})| \, dx \, dt = \varepsilon(n,j,\mu)$$

On the other hand,the second tem of the right-hand side of (1.21) reads as

$$\frac{\rho(m)}{\alpha} \int_Q a(x,t,T_m(u_n),\nabla T_m(u_n)) \, \nabla T_m(u_n) |\psi(T_m(u_n) - \kappa^i_{\mu,j})| \, dx \, dt$$

$$= \frac{\rho(m)}{\alpha} \int_Q \big[a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(v_j)\chi^s_j)\big]$$

$$\times \big[\nabla T_m(u_n) - \nabla T_m(v_j)\chi^s_j\big] \big|\psi\big(T_m(u_n) - \kappa^i_{\mu,j}\big)\big| \, dx \, d$$

$$+\frac{\rho(m)}{\alpha}\int_Q a\big(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s\big)\,[\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]$$
$$|\psi(T_m(u_n)-\kappa_{\mu,j}^i)|\;dx\;dt$$
$$\frac{\rho(m)}{\alpha}\int_Q a(x,t,T_m(u_n),\nabla T_m(u_n))\,\nabla T_m(v_j)\chi_j^s|\psi(T_m(u_n)-\kappa_{\mu,j}^i)|\;dx\;dt$$

**A**s above ,by letting successively first $n$,than $j,\mu$ and finally $s$ go to infinity ,we can easily see that each one of last two integrals of the right-land side of thelast equality is of the form $\varepsilon(n,j,\mu)$ and then
un≤mgnx,t,un,∇unψTmun-κμ,ji dx dt

$$\left|\iint_{|u_n|<m} g_n\,(x,t,u_n,\nabla u_n)\psi\big(T_m(u_n)-\kappa_{\mu,j}^i\big)\;dx\;dt\right|$$
$$+\left|\iint_{|u_n|<m} H_n\,(x,t,,\nabla u_n)\psi\big(T_m(u_n)-\kappa_{\mu,j}^i\big)\;dx\;dt\right|$$

$$\leq\frac{\rho(m)}{\alpha}\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(v_J)\chi_j^s)]$$
$$\times\left[\nabla T_m(u_n)-\nabla T_m\big(v_j\big)\chi_j^s\right]\left|\psi\big(T_m(u_n)-\kappa_{\mu,j}^i\big)\right|\;dxdt+\varepsilon(n,j,\mu)\qquad(3.22)$$

. By combining (3.16),(3.17),(3.19),(3.20)and (3.22)we get

$$\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s)][\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]$$
$$\times\left[\psi'(T_m(u)-\kappa_{\mu,j}^i)-\frac{\rho(m)}{\alpha}\left|\eta\big(T_m(u_n)-\kappa_{\mu,j}^i\big)\right|\right]dxdt\leq\varepsilon(n,j,\mu,i,s)$$

and so, thanks to (3.14)

$$\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s)]$$
$$\times[\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]dxdt\leq 2\varepsilon(n,j,\mu,i,s)\qquad(3.23)$$

On the other hand ,we have

$$\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(u)\chi_j^s)][\nabla T_m(u_n)-\nabla T_m(u)\chi^s]dxdt$$
$$-\int_Q[\,a(T_m(u_n),\nabla T_m(u_n))-a(T_m(u_n),\nabla T_m(v_j)\chi_j^s)][\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]dxdt$$
$$=\int_Q a\,(x,t,T_m(u_n),\nabla T_m(u_n))[\nabla T_m(v_j)\chi_j^s-\nabla T_m(u)\chi^s]dxdt$$
$$-\int_Q a\,(x,t,T_m(u_n),\nabla T_m(u)\chi^s)[\nabla T_m(u_n)-\nabla T_m(u)\chi^s]dxdt$$
$$+\int_Q a\,(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s)[\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]dxdt$$

and ,as it can be easily seen ,each integral of the right-hand side is of the form$\varepsilon(n,j,s)$,implying that

$$\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(u)\chi^s)][\nabla T_m(u_n)-\nabla T_m(u)\chi^s]dxdt$$
$$=\int_Q[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s)]$$
$$\times[\nabla T_m(u_n)-\nabla T_m(v_j)\chi_j^s]dxdt+\varepsilon(n,j,s).\qquad(3.24)$$

For $r\leq s$, we have

$$0\leq\int_{Q^r}[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(u))][\nabla T_m(u_n)-\nabla T_m(u)]dxdt$$
$$\leq\int_{Q^s}[\,a(x,t,T_m(u_n),\nabla T_m(u_n))-a(x,t,T_m(u_n),\nabla T_m(u))][\nabla T_m(u_n)-\nabla T_m(u)]dxdt$$

$$= \int_{Q^s} [a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(u)\chi^s)][\nabla T_m(u_n) - \nabla T_m(u)\chi^s]dxdt$$
$$\le \int_{Q} [a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(u)\chi^s)][\nabla T_m(u_n) - \nabla T_m(u)\chi^s]dxdt$$
$$= \int_{Q} [a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(v_j)\chi_j^s)][\nabla T_m(u_n) - \nabla T_m(u)\chi_j^s]dxdt + \varepsilon(n,j,s)$$
$$\le \varepsilon(n,j,\mu,i,s),$$
hence by passing to the limit sup over $n$,get
$$0 \le \lim_{n\to\infty} sup \int_{Q} [a(x,t,T_m(u_n),\nabla T_m(u_n)) - a(x,t,T_m(u_n),\nabla T_m(u))][\nabla T_m(u_n) - \nabla T_m(u)]\, dxdt$$
$$\le \lim_{n\to\infty} sup\ \varepsilon(n,j,\mu,i,s)$$
in which we let successively $j \to \infty$ , $\mu \to \infty$ $i \to \infty$ and $s \to \infty$ to obtain
$$\int_{Q} [a(T_m(u_n),\nabla T_m(u_n)) - a(T_m(u_n),\nabla T_m(u))]$$
$$[\nabla T_m(u_n) - \nabla T_m(u)]\, dxdt \to 0 \text{ as } n \to \infty$$
and thus,as in the elliptic case (see ), there exists a subsequence also denoted by $u_n$ such that
$$\nabla u_n \to \nabla u \ \ a.e.\ in\ Q. \qquad (3.25)$$
We then deduce that, for all $m > 0$ $a(x,t,T_m(u_n),\nabla T_m(u_n))) \rightharpoonup a(x,t,T_m(u),\nabla T_m(u))$
and $a(x,t,u_n,\nabla u_n) \rightharpoonup a(x,t,u,\nabla u)$ weakly in $(L_\psi(Q))^N for \sigma(\prod L_\psi,\prod E_\varphi)$.
Thanks to(3.23) and (3.24),we can write
$$\int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt$$
$$\le \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u)\chi^s dxdt$$
$$+ \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n)\chi^s)[\nabla T_m(u_n) - T_m(u)\chi^s]dxdt + \varepsilon(n,j,\mu,i,s)$$
and than
$$\lim_{n\to\infty} sup \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt$$
$$\le \int_{Q} a\,(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)\chi^s dxdt$$
$$+ \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u)\chi^s)[1-\chi^s]dxdt$$
$$+ \lim_{n\to\infty} \varepsilon(n,j,\mu,i,s),$$
in which we can pass to the limit as $j,\mu,i,s \to \infty$ to obtain
$$\lim_{n\to\infty} sup \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt \le \int_{Q} a\,(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt$$
On the other hand ,Fatou’s lemma implies
$$\int_{Q} a\,(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt \le \lim_{n\to\infty} inf \int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt$$
and thus, as $n \to \infty$,
$$\int_{Q} a\,(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt \to \int_{Q} a\,(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt$$
Since $a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n) \ge d(x,t) \in L^1(Q)$ we deduce that
$$a(x,t,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt \to a(x,t,T_m(u),\nabla T_m(u))\nabla T_m(u)dxdt\ in L^1(Q), \qquad (3.26)$$
as $n \to \infty$; implying by using (3.3) and Vitali’s theorem that
$\nabla T_m(u_n) \to \nabla T_m(u)$ in $(L_\varphi(Q))^N$ for the modular convergence.

**Step 3: equi-integrability of the nonlinearities**

We shall now prove that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ and $H_n(x,t,\nabla u_n) \to H(x,t,\nabla u)$ strongly in $L^1(Q)$ by using Vitali's theorem. Since $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ and $H_n(x,t,\nabla u_n) \to H(x,t,\nabla u)$ a.e. in Q,thanks to (3.12) and (3.24),it suffices to $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in Q. Let $E \in Q$ be a measurable subset of Q. We have for any $m > 0$

$$\int_E |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|\ \ dxdt = \int_{E\cap\{|u_n|\le m\}} |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt + \int_{E\cap\{|u_n|>m\}} |\ g_n(x,t,u_n,\nabla u_n)dxdt + H_n(x,t,\nabla u_n)|\ \ dxdt$$

On the one hand

$$\int_{E\cap\{|u_n|>m\}} |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt \le \frac{1}{m}\int_Q |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|u_n dxdt \le \frac{D}{m}$$

where $D$ is the constant in (3.12). Therefore , there exists $m = m(\varepsilon)$ large enough such that

$$\int_{E\cap\{|u_n|>m\}} |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt \le \frac{\varepsilon}{2}\ \forall n$$

On the other hand

$$\int_{E\cap\{|u_n|\le m\}} |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt$$
$$\le \int_E |\ g_n(x,t,T_m(u_n),\nabla T_m(u_n)) + H_n(x,t,T_m(u_n),\nabla T_m(u_n))|dxdt$$
$$\le \rho(m)\int_E [\ d_2(x,t) + \varphi(x,|\nabla T_m(u_n)|)]dxdt$$
$$\le \rho(m)\int_E [\ d_2(x,t) + \frac{1}{\alpha}d(x,t)|)]dxdt + \frac{\rho(m)}{\alpha}\int_E a\ (T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)dxdt$$

By virtue of strong convergence (3.26) and the fact that $d_2(x,t), d(x,t) \in L_1(Q)$, there exists $\nu$ suth that

$$|E| < \nu \Rightarrow \int_{E\cap\{|u_n|>m\}} |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt \le \frac{\varepsilon}{2}\forall n$$

Consequently

$$|E| < \nu \Rightarrow \int_E |\ g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)|dxdt \le \varepsilon\forall n$$

which shows that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in Q as required.

**Step 4:Passage to the limit and regularity of the solution**

Let $v \in W_0^{1,x}L_\varphi(Q) \cap L^\infty(Q)$such that$\frac{\partial v}{\partial t} \in W^{-1,x}L\psi(Q) + L^2(Q)$. There exists a prolongation $\bar{v}$ of $v$ such that (see proof of Lemmal)

$$\bar{v} = v\ on\ Q, \bar{v} \in W_0^{1,x}L_\varphi(\Omega\times R) \cap L^2(\Omega\times R) \cap L^\infty(\Omega\times R),$$

and

$$\frac{\partial\bar{v}}{\partial t} = v \in W^{-1,x}L\psi(\Omega\times R) + L^2(\Omega\times R). \qquad (3.27)$$

By Theorem 1(see also Remark1), there exists asequence $(\kappa_j \subset D(\Omega\times R))$ such that

$$\kappa_j \to \bar{\upsilon} \; in \; W_0^{1,x} L_\varphi(\Omega \times R) \cap L^2(\Omega \times R)$$

and

$$\frac{\partial \kappa_j}{\partial t} \to \frac{\partial \bar{\upsilon}}{\partial t} \; in \; W^{-1,x} L\psi(\Omega \times R) + L^2(\Omega \times R), \qquad (3.28)$$

For the modular convergence and $\| \kappa_j \|_{\infty,\Omega\times R} \leq (N+2) \| \bar{\upsilon} \|_{\infty,\Omega\times R}$. Go back to approximate equations (3.11) and use $\kappa_j \chi_(0,\tau)$, for every $\tau \in [0,T]$ (which belongs to $W_0^{1,x} L_\varphi(Q)$) as a test funcion one has

$$\langle \frac{\partial u_n}{\partial t}, \kappa_j \rangle Q_\tau + \int_{Q_\tau} a\,(x,t,u_n,\nabla u_n)\nabla \kappa_j \, dxdt$$
$$+ \int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)\kappa_j \, dxdt + \int_{Q_\tau} H_n\,(x,t,\nabla u_n)\kappa_j \, dxdt = \langle f, \kappa_j \rangle Q_\tau. \quad (3.29)$$

which implies that

$$[\int_Q u_n\,(t)\kappa_j(t)dx]_0^T - \int_{Q_\tau} u_n \frac{\partial \kappa_j(t)}{\partial t} \, dx\, dt + \int_{Q_\tau} a\,(x,t,u_n,\nabla u_n)\nabla \kappa_j \, dx\, dt$$
$$+ \int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)\kappa_j \, dx\, dt + \int_{Q_\tau} H_n\,(x,t,\nabla u_n)\kappa_j \, dx\, dt = \langle f, \kappa_j \rangle_{Q_\tau}$$

We shall go to the limit as $n \to \infty$ in all terms of (3.29).Since for all$j, \kappa\chi_(0,\tau) \in D(\bar{Q}_\tau)$) we have

$$-\int_{Q_\tau} u_n \frac{\partial \kappa_j(t)}{\partial t} dxdt \to -\int_{Q_\tau} u \frac{\partial \kappa_j(t)}{\partial t} dxdt,$$
$$\int_{Q_\tau} a\,(x,t,u_n,\nabla u_n)\nabla \kappa_j \, dxdt \to \int_{Q_\tau} a\,(x,t,u,\nabla u)\nabla \kappa_j \, dxdt$$

and

$$\int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)\kappa_j \, dx\, dt \to \int_{Q_\tau} g\,(x,t,u,\nabla u)\kappa_j \, dx\, dt$$
$$\int_{Q_\tau} H_n\,(x,t,\nabla u_n)\kappa_j \, dx\, dt \to \int_{Q_\tau} H\,(x,t,\nabla u)\kappa_j \, dx\, dt$$

To go to the limit as $n \to \infty$ in the fist term of (3.29), we will first prove that $u_n \to u$ in $C([0,T], L^2(\Omega))$ (implying in particular ,that $u \in C([0,T], L^2(\Omega))$ ). To do that ,let now $\kappa_{j,\mu}^{i,l} = T_l(v_j)_\mu + exp(-\mu t) T_l(\kappa_i)$ and $\kappa_{j,\mu}^{i,l} = T_l(u_j)_\mu + exp(-\mu t) T_l(\kappa_i)$,for every $l > 0$.On one hand ,we have for every $\tau \in (0,T]$

$$\langle (\kappa_{j,\mu}^{i,l})', u_n - \kappa_\mu^{i,l} )\rangle_{Q_\tau} = \mu \int_{Q_\tau} (T_l)(v_j) - \kappa_{j,\mu}^{i,l}))(u_n - \kappa_{j,\mu}^{i,l})dxdt$$
$$\to \mu \int_{Q_\tau} (T_l)(v_j) - \kappa_{j,\mu}^{i,l}))(u - \kappa_{j,\mu}^{i,l})dxdt$$
$$\to \mu \int_{Q_\tau} (T_l)(u) - \kappa_{j,\mu}^{i,l}))(u - \kappa_{j,\mu}^{i,l})dxdt \geq 0 \qquad (3.30)$$

as first $n \to \infty$ and than $j \to \infty$ and where we have used the fact that $\kappa_\mu^{i,l} \leq l$ to get the positiveness of last integral. On the other hand ,by using (3.11)

$$\langle u_n', u_n - \kappa_{j,\mu}^{i,l} \rangle_{Q_\tau} = \langle f, u_n - \kappa_{j,\mu}^{i,l} \rangle_{Q_\tau} + \int_{Q_\tau} a\,(u_n,\nabla u_n)[\nabla \kappa_{j,\mu}^{i,l} - \nabla u_n]\, dxdt$$
$$+ \int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)(\kappa_{j,\mu}^{i,l} - u_n)\, dx\, dt \int_{Q_\tau} H_n\,(x,t,\nabla u_n)(\kappa_{j,\mu}^{i,l} - u_n)\, dx\, dt$$

in which we can use Fatou's Lemma and Lebesgue to pass to the limit sup first over n and $j, \mu, l$, to get

$$\langle u_n', u_n - \kappa_{J,\mu}^{i,L} \rangle_{Q_\tau} \leq \varepsilon(n,j,\mu,l) \; not \; depending \; on \; \tau \qquad (3.31)$$

Therefore, by writing

$$\frac{1}{2} \left\| u_n(\tau) - \kappa_{J,\mu}^{i,L}(\tau) \right\|_{L^2(\Omega)}^2 = \langle u_n' - (\kappa_{J,\mu}^{i,L})', u_n - \kappa_{J,\mu}^{i,L} \rangle_{Q_\tau}$$
$$+ \frac{1}{2} \int_{Q_\tau} (\, u_0 - T_l(\kappa_i))^2 dxdt$$

$$= \langle u_n' - (\kappa_{J,\mu}^{i,L})', u_n - \kappa_{J,\mu}^{i,L} \rangle_{Q_\tau} + \frac{1}{2} \left\| u_n(\tau) - \kappa_{J,\mu}^{i,L}(\tau) \right\|_{L^2(\Omega)}^2$$

and using (3.30) and (3.31),we deduce that $\left\| u_n(\tau) - \kappa_{J,\mu}^{i,L}(\tau) \right\|_{L^2(\Omega)}^2 \leq \varepsilon(n,j,\mu,l,i)$ not depending $\tau \in (0,T]$.This implies that

$$\| u_n(\tau) - u_m(\tau) \|_{L^2(\Omega)} \leq \varepsilon(n,m)$$

not depending on $\tau \in [0,T]$, and thus, $u_n$ is a Cauchy sequence in $C([0,T], L^2(\Omega))$. Since the limit of $u_n$ in $L^1(Q)$ is u we deduce that

$u_n \to u$ in $C([0,T], \Omega)$,

therefore, by letting $n \to \infty$ in the first term of (3,29), we have

$$[\int_\Omega u_n\,(t)\ \kappa_j(t)\ dx]_0^\tau \to [\int_\Omega u\,(t)\ \kappa_j(t)\ dx]_0^\tau.$$

Consequently, by letting $n \to \infty$ in (3.28),we get

$$[\int_\kappa u\,(t)\ \kappa_j(t)\ dx]_0^\tau - \int_{Q_\tau} u\ \frac{\partial \kappa_j}{\partial t}\ dx\ dt + \int_{Q_\tau} a\,(x,t,u,\nabla u) \nabla \kappa_j\ dx\ dt$$
$$+ \int_{Q_\tau} g\,(x,t,u,\nabla u) \kappa_j\ dx\ dt + \int_{Q_\tau} H\,(x,t,\nabla u) \kappa_j\ dx\ dt = \langle f, \kappa_j \rangle_{Q_\tau}. \qquad (3.32)$$

we shall now go to the limit as $j \to \infty$ in all terms of (3.32).In view of (3.31)and the fact that $\kappa_j$ are uniformly bounded ,there is no problem to pass to the limit in last four terms of (3.32).For what concerns the first one,observe that ,as in the proof of lemma 3,4 we have $\kappa_j \to v$ in $C([0,T], L^2(\Omega))$. Therefore, we can let $j \to \infty$ in all terms of (3.32) to get

$$[\int_\kappa u\,(t)\ v(t)\ dx]_0^\tau - \langle \frac{\partial v}{\partial t}, u \rangle_{Q_\tau} + \int_{Q_\tau} a\,(x,t,u,\nabla u) \nabla v\ dx\ dt$$
$$+ \int_{Q_\tau} g\,(x,t,u,\nabla u) v\ dx\ dt = \langle f, v \rangle_{Q_\tau},$$

which shows that u satisfies all properties of Theorem 5.1 It only remains to prove the energy equality. For that, we use, for a given $m\textgreater 0,T_{m}(u_{n})$, as a test function in(3,11).to get

$$\langle u_n', T_m(u_n) \rangle_{Q_\tau} = -\int_{Q_\tau} a\,(x,t,u_n,\nabla u_n) \nabla T_m(u_n)\ dx\ dt$$
$$-\int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n)\ T_m(u_n)\ dx\ dt - \int_{Q_\tau} H_n\,(x,t,\nabla u_n)\ T_m(u_n)\ dx\ dt + \langle f, T_m(u_n) \rangle_{Q_\tau},$$

which gives by setting $S_m(s) = \int_0^s T_m\,(z) dz$,

$$\int_\Omega S_m\,(u_n(\tau))\ dx - \int_\Omega S_m\,(u_0)\ dx =$$
$$-\int_{Q_\tau} a\,(x,t,u_n,\nabla u_n) \nabla T_m(u_n)\ dx\ dt - \int_{Q_\tau} g_n\,(x,t,u_n,\nabla u_n) T_m(u_n)\ dx\ dt$$
$$-\int_{Q_\tau} H_n\,(x,t,\nabla u_n) T_m(u_n)\ dx\ dt + \langle f, T_m(u_n) \rangle_{Q_\tau}, \qquad (3.33)$$

Recall that $| \int_\Omega S_m\,(u_n(\tau)) | \leq m|u_n(\tau)| \longrightarrow m|u(\tau)|$ in $L^2(\Omega)$ as $n \to \infty$, then, by using Lebesgue theorem and(3,26) ,we can pass to the limit as $n \to \infty$ each term of (1,33) to obtain

$$\int_\Omega S_m\,(u\tau)\ dx - \int_\Omega S_m\,(u_0)\ dx =$$
$$-\int_{Q_\tau} a\,(x,t,u,\nabla u) \nabla T_m(u)\ dx\ dt - \int_{Q_\tau} g\,(x,t,u,\nabla u) T_m(u)\ dx\ dt$$
$$-\int_{Q_\tau} H\,(x,t,\nabla u) T_m(u)\ dx\ dt + \langle f, T_m(u) \rangle_{Q_\tau}. \qquad (3.34)$$

Observe that for every $s \in R$, $|S_m(s)| \leq \frac{s^2}{2}$ and $S_m(s) \to \frac{s^2}{2}$ as $m \to \infty$, so that, by using Lebesgue theorem and the fact that $u(\tau) \in L^2(\kappa)$,we have, as $m \to \infty$,

$\int_\kappa S_m\,(u(\tau))\,dx \to \frac{1}{2}\int_\kappa u^2\,(\tau)$ and $\int_\kappa S_m\,(u_0)\,dx \to \frac{1}{2}\int_\kappa S_m\,(u_0)^2\,dx.$
Remark also that

$$|\int_{Q_\tau} a\,(x,t,T_m(u)\,,\nabla T_m(u)\,)\nabla T_m(u)| \le a(x,t,u,\nabla u)\nabla u \in L^1(Q)$$

and

$$|\int_{Q_\tau} g\,(x,t,T_m(u)\,,\nabla T_m(u)\,)\nabla T_m(u)| \le g(x,t,u,\nabla u)\nabla u \in L^1(Q)$$

and

$$|\int_{Q_\tau} H\,(x,t,T_m(u)\,,\nabla T_m(u)\,)\nabla T_m(u)| \le H(x,t,u,\nabla u)\nabla u \in L^1(Q)$$

therefore, it is easy to pass the limit as $m \to \infty$ in (3,34) to get the energy equality

$$[\frac{1}{2}\int_\kappa u\,(t)^2\,dx]_0^\tau + \int_{Q_\tau} a\,(x,t,u,\nabla u)\nabla u\,dx\,dt + \int_{Q_\tau} g\,(x,t,u,\nabla u)u\,dx\,dt + \int_{Q_\tau} H\,(x,t,\nabla u)u\,dx\,dt = \langle f,u\rangle_{Q_\tau}.$$

This completes the proof of Theorem 3.1

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